SIMPLE CURVES IN $\mathbb{R}^n$ AND AHLFORS' SCHWARZIAN DERIVATIVE

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Abstract

We derive sharp injectivity criteria for mappings $f : (-1,1) \to \mathbb{R}^n$ in terms of Ahlfors' definition of the Schwarzian derivative for such mappings.

1. INTRODUCTION

Because the Schwarzian derivative $Sf = (f''/f')' - \frac{1}{2}(f''/f')^2$ measures the extent to which an analytic function deviates from being a Möbius transformation, it carries information about both the local and global behavior of conformal mappings. Although in regard to the former, Sf says something about how f alters cross-ratios and curvature, the importance it has acquired in geometric function theory and related areas over the last 50 years or so stems primarily from Nehari's fundamental papers [Ne 1], [Ne 2] on univalence criteria of the form

$$|Sf(z)| \le 2P(|z|) \tag{1.1}$$

for analytic functions f in the unit disk. In his most general version of this criterion [Ne 2], P can be any even function for which (i) $(1-x^2)^2 P(x)$ is non-increasing on $[0,1)$, and (ii) the even solution of $U'' + PU = 0$ has no zeros. It is a straightforward consequence of condition (i) that (1.1) will imply univalence for any P for which

$$\varphi : (-1,1) \to \mathbb{C} \text{ and } |S\varphi(x)| \le 2P(|x|) \;\Rightarrow\; \varphi \,\text{is injective}\,, \tag{1.2}$$

so that the matter reduces in essence to showing that (1.2) holds under assumption (ii).

In this paper we shall give a very short proof that a stronger form of (1.2) actually holds under a weaker assumption on P, and more importantly, that such injectivity criteria hold for $f : (-1,1) \to \mathbb{R}^n$. In this wider context of curves in space we use a corresponding version of the Schwarzian due to Ahlfors [Ah], for which we offer a geometrically appealing definition, rather different in tenor from his, and which makes manifest that in this extended context, Sf continues to be a complex number invariant under Möbius transformations. Our analysis of the injectivity of f and of the related issues of continuous extendibility to $[-1,1]$ and extremal behavior is based

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largely on an observation implicit in [Ne 2] to the effect that it is only the real part of Sf that is of significance in such questions.

2. HIGHER DIMENSIONAL CURVES

In [Ah] Ahlfors generalized the Schwarzian to cover $f : (a,b) \to \mathbb{R}^n$ by separately defining analogues of the 2-dimensional $\Re\{Sf\}$ and $\Im\{Sf\}$ as

$$S_1 f = \frac{\langle f', f''' \rangle}{|f'|^2} - 3\frac{\langle f', f'' \rangle^2}{|f'|^4} + \frac{3}{2}\frac{|f''|^2}{|f'|^2}\,, \tag{2.1}$$

and

$$S_2 f = \frac{f' \wedge f'''}{|f'|^2} - 3\frac{\langle f', f'' \rangle}{|f'|^4} f' \wedge f''\,, \tag{2.2}$$

respectively. Here, for $\vec{a}, \vec{b} \in \mathbb{R}^n$, $\vec{a} \wedge \vec{b}$ is the antisymmetric bivector with components $(\vec{a} \wedge \vec{b})_{ij} = a_i b_j - a_j b_i$ and norm $(\sum_{i<j}(a_i b_j - a_j b_i)^2)^{1/2}$. Ahlfors indicated that he was led to these seemingly esoteric definitions by a direct identification of $\Re\{z\overline{\zeta}\}$ with the inner product $\langle z, \zeta \rangle$ of the 2-dimensional vectors z, ζ and the far from obvious identification of $\Im\{z\overline{\zeta}\}$ with the corresponding $\zeta \wedge z$ based on the fact that $(\Im\{z\overline{\zeta}\})^2 = |\zeta \wedge z|^2$. In this section we give an equivalent but geometrically convincing derivation of what amounts to Alhfors' Schwarzian, very much in the spirit of his definition of the complex cross-ratio of four points in $\mathbb{R}^n$.

Let C be a curve in $\mathbb{R}^n$, $n \geq 3$, parametrized by the C^3 function f on (a,b) with nonvanishing f'. It is well-known that for each $t_0 \in (a,b)$ on C, there is a C^∞ function $g : (a,b) \to \mathbb{R}^n$ and a 2-sphere $K(t_0)$ (the osculating 2-sphere, which can degenerate into a plane; see, e.g., [L]) such that

$$g((a,b)) \subset K(t_0)$$

and

$$f(t) = g(t) + o(|t - t_0|^3)\,,\ t \to t_0\,. \tag{2.3}$$

By regarding $K(t_0)$ as $\mathbb{C}$ via a stereographic projection, one can identify g with a $\phi : (a,b) \to \mathbb{C}$, for which the expression $S\phi = (\phi''/\phi')' - (1/2)(\phi''/\phi')^2$ of Section 2 is meaningful. In the case of a nondegenerate osculating sphere, one can take the vector from the point of contact to the center as $(0,0,R), R > 0$ and give to the tangent plane, our $\mathbb{C}$, its usual (to be referred to as "canonical" below) orientation as $\mathbb{C} = \mathbb{R}^2 \subset \mathbb{R}^3$. At points at which the osculating sphere degenerates to a plane, however, there is no canonical orientation for this plane, nor is there any canonical copy of $\mathbb{R}^3$ containing this plane. To circumvent this inherent ambiguity, we shall define $\mathcal{S}f(t_0)$ to be $S\phi(t_0)$ or $\overline{S\phi(t_0)}$, whichever one has nonnegative imaginary part. Indeed, this is consist with the cross-ratio $(\vec{a}, \vec{b}, \vec{c}, \vec{d})$ of $\vec{a}, \vec{b}, \vec{c}, \vec{d} \in \mathbb{R}^n$ as defined by Ahlfors in [Ah]: any given four points are always contained in a (possibly degenerate) 2-sphere K . One regards K as $\mathbb{C}$, calculates the usual cross-ratio k, and gives to $(\vec{a}, \vec{b}, \vec{c}, \vec{d})$ the value k or $\bar{k}$, whichever has nonnegative imaginary part.

We show that $\mathcal{S}f(t_0) = S_1 f(t_0) + i|S_2 f(t_0)|$, thereby justifying the contention that the single complex number $\mathcal{S}f(t_0)$ embodies all of the information carried by Ahlfors' 2-part Schwarzian. We first consider the case of a nondegenerate osculating sphere. First of all, it is clear that both $S_1 f(t_0)$ and $|S_2 f(t_0)|$ remain unchanged when f is replaced by $\rho U f + \vec{c}$, where $\rho \in \mathbb{R}\backslash\{0\}$, U is a proper orthogonal transformation of $\mathbb{R}^n$, and $\vec{c} \in \mathbb{R}^n$ is a constant. Thus we may limit ourselves to the case in which $K(t_0)$ is the 2-sphere contained in $\mathbb{R}^3 = \{(x_1, x_2, x_3, 0, \ldots, 0) : x_1, x_2, x_3 \in \mathbb{R}\}$ with center at $(0,0,1)$. We denote by

$$P(x+iy) = \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, 1 - \frac{2}{1+x^2+y^2} \right)$$

the inverse of the usual stereographic projection of $\mathbb{C}$ onto the sphere in $\mathbb{R}^3$. In $\mathbb{R}^3$ the components of $\vec{a} \wedge \vec{b}$ are effectively those of $\vec{a} \times \vec{b}$. Let $h(t) = x(t) + iy(t)$, with $h(0) = 0$. A straightforward, somewhat tedious calculation shows that

$$S_1(P \circ h)(0) = \Re\{\mathcal{S}h(0)\} \tag{2.4}$$

and

$$S_2(P \circ h)(0) = (0, 0, \Im\{\mathcal{S}h(0)\})\,. \tag{2.5}$$

In fact, these relations can be verified easily with any symbolic manipulation program, such as Maple or Mathematica, or even on a TI-92 calculator, since one can limit consideration to the case that x and y are cubic polynomials in t. From this the desired relation, $\mathcal{S}f(t_0) = S_1 f(t_0) + i|S_2 f(t_0)|$, follows immediately. In the case that the osculating sphere degenerates to a plane, by appropriate choices of ρ, U and $\vec{c}$, we can arrange for this plane to be $\mathbb{R}^2 = \{(x_1, x_2, 0, \ldots, 0) : x_1, x_2 \in \mathbb{R}\}$. Relations (2.4) and (2.5) again follow either by a limit argument or by direct calculation. We stress that in both cases the exact choice of g is irrelevant since, in light of (2.3), only derivatives of order up to 3 enter into the calculations.

Theorem A: *Let $f : (a, b) \to \mathbb{R}^n$ be a C^3 curve with nowhere vanishing f'.*

(i) For any Möbius transformation T of $\mathbb{R}^n$, $\mathcal{S}(T \circ f) = \mathcal{S}f$.

(ii) $(f(t_0 + t\alpha), f(t_0 + t\beta), f(t_0 + t\gamma), f(t_0 + t\delta)) =$

$$(\alpha, \beta, \gamma, \delta)[1 + \frac{1}{6}(\alpha - \beta)(\gamma - \delta)\mathcal{S}^* f(t_0)t^2] + o(t^2)\,, \text{ as } t \to 0\,,$$

where $\mathcal{S}^* f$ is $\mathcal{S}f$ or its conjugate according to whether $(\alpha, \beta, \gamma, \delta)(\alpha - \beta)(\gamma - \delta)$ is nonnegative or not.

Comment: Conclusion (i) implies that $\mathcal{S}f$ has meaning for C^3 mappings $f : (a, b) \to \mathbb{R}^n \cup \{\infty\} = \mathbb{S}^n \subset \mathbb{R}^{n+1}$. Conclusion (ii) extends a similar relation involving Ahlfors' $S_1 f = \Re\{\mathcal{S}f\}$.

Proof: (i) For $t_0 \in (a, b)$ let $K(t_0)$ be the corresponding osculating sphere and let $g = g(t)$ be as in (2.3). The Möbius transformation T will carry $K(t_0)$ onto the osculating 2-sphere of $T \circ f$ at $T \circ f(t_0)$, at which point this curve has contact of order 3 with $T \circ g$. According to our definition, $\mathcal{S}f(t_0)$ and $\mathcal{S}(T \circ f)(t_0)$ are interpreted as complex numbers after stereographically projecting the respective curves g and $T \circ g$ onto the complex plane. Because T is Möbius, it is clear that the two stereographic projections are related by a planar Möbius mapping, which will preserve the Schwarzian as defined.

(ii) To show this, observe that the relevant terms in the expansion considered will remain unchanged if we replace f with g. After a suitable stereographic projection of the curve given by g, we can assume that we are working in $\mathbb{C}$. This formula is valid with f replaced by g and $\mathcal{S}^* f$ by Sg. The desired conclusion now follows by replacing the imaginary parts on both sides by their absolute values.

Going back to relations (2.1) and (2.2), $S_1 f$ and $S_2 f$ can be written in terms of the geometry of the trace of f. We write

$$f' = v\hat{t} \quad \text{and} \quad f'' = v'\hat{t} + v^2 k\hat{n}\,,$$

where $v > 0$ and $\hat{t}, \hat{n}$ are the unit tangent and normal vectors. A third differentiation gives

$$f''' = v''\hat{t} + vv'k\hat{n} + 2vv'k\hat{n} + v^2k'\hat{n} + v^2k\hat{n}'\,.$$

Since $\hat{n}$ is a unit vector, $\langle \hat{n}', \hat{n} \rangle = 0$, and upon differentiating $\langle \hat{t}, \hat{n} \rangle = 0$ we see that the component of $\hat{n}'$ in the direction of $\hat{t}$ must equal $-vk$. Thus the equation

$$\hat{n}' = -vk\hat{t} + v\tau\hat{b}$$

defines both the binormal vector $\hat{b}$ and the torsion τ. From this we obtain

$$f''' = (v'' - v^3k^2)\hat{t} + (3vv'k + v^2k')\hat{n} + v^3k\tau\hat{b}\,,$$

so that

$$S_1 f = \frac{v'' - v^3k^2}{v} - 3\frac{(v')^2}{v^2} + \frac{3}{2}\frac{(v')^2 + v^4k^2}{v^2} = \left(\frac{v'}{v}\right)' - \frac{1}{2}\left(\frac{v'}{v}\right)^2 + \frac{1}{2}v^2k^2\,.$$

Thus, if $s(x)$ denotes arc length, then

$$S_1 f = Ss(x) + \frac{1}{2}v^2k^2\,. \tag{2.6}$$

Although it will not be used in the sequel, we derive a corresponding formula for $S_2 f$. It follows from the expressions given above for f', f'' and f''' that

$$f' \wedge f'' = v^3k(\hat{t} \wedge \hat{n}) \quad \text{and} \quad f' \wedge f''' = v^2(3v'k + vk')(\hat{t} \wedge \hat{n}) + v^4k\tau(\hat{t} \wedge \hat{b})\,.$$

A computation gives that

$$\langle \vec{a} \wedge \vec{b}, \vec{a} \wedge \vec{c} \rangle = |\vec{a}|^2 \langle \vec{b}, \vec{c} \rangle - \langle \vec{a}, \vec{b} \rangle \langle \vec{a}, \vec{c} \rangle\,,$$

which implies that in the $n(n-1)/2$ dimensional space, $\hat{t} \wedge \hat{n}$ and $\hat{t} \wedge \hat{b}$ are orthonormal. With this we now write

$$S_2 f = (3v'k + vk')(\hat{t} \wedge \hat{n}) + v^2k\tau(\hat{t} \wedge \hat{b}) - 3v'k(\hat{t} \wedge \hat{n}) = vk'(\hat{t} \wedge \hat{n}) + v^2k\tau(\hat{t} \wedge \hat{b})\,.$$

3. INJECTIVITY CRITERIA AND EXTENDIBILITY

In several places in the proofs to follow, we make use of the classical Strum comparison theorem, which we state here for reference.

Theorem: *Let u, v be positive functions on (a, b) which satisfy $u'' + pu = 0$, $v'' + qv = 0$, where $p \le q$, and $u(x_0) = v(x_0)$, $u'(x_0) = v'(x_0)$ for some $x_0 \in (a, b)$. Then $u \ge v$ on (a, b).*

For convenience, we use Ahlfors' original notation $S_1 f$ for $\Re\{\mathcal{S}f\}$.

Theorem B: *Let $P = P(x)$ be a continuous function defined on $(-1, 1)$ with the property that no non-trivial solution u of $u'' + Pu = 0$ has more than one zero. Let $f : (-1, 1) \to \mathbb{R}^n \cup \{\infty\}$ be a curve of class C^3 with nowhere vanishing f'. If $S_1 f(x) \le 2P(x)$ on $(-1, 1)$ then f is one-to-one.*

Proof: If not, then $f(x_1) = f(x_2)$ for $x_1 < x_2$ in $(-1, 1)$, where f is one-to-one on $[x_1, x_2)$. Let $g = T \circ f$ be a Möbius transformation of f that takes $f(x_1)$ to the point at infinity, and let $v = |g'|^{-1/2}$. Then v is regular in the open interval (x_1, x_2), and a simple calculation shows that $v'' + qv = 0$, where

$$2q = \frac{\langle g', g''' \rangle}{|g'|^2} + \frac{|g''|^2}{|g'|^2} - \frac{5}{2}\frac{\langle g', g'' \rangle^2}{|g'|^4} = S_1 g - \frac{1}{2}\left(\frac{|g''|^2}{|g'|^2} - \frac{\langle g', g'' \rangle^2}{|g'|^4}\right) \le S_1 f\,, \tag{3.1}$$

hence $q \leq P$. A suitable solution U_1 of $U'' + PU = 0$ coincides with v to first order at some point $x_0 \in (x_1, x_2)$, so that by the Sturm comparison theorem, $v(x) \geq U(x)$ on (x_1, x_2). Since by hypothesis U has at most one zero in the interval $(-1, 1)$, we conclude that v has a positive lower bound in a neighborhood of either x_1 or x_2. But then $|g'|$ will be bounded above in that neighborhood, making it impossible for g to become infinite there.

In light of (2.6) we have

Corollary C: *Let P be as in the previous theorem and let $f : (-1, 1) \to \mathbb{R}^n$ be an arclength parametrized curve with geodesic curvature k. If $k^2(s) \leq 4P(s)$ on $(-1, 1)$ then f is one-to-one.*

Interesting examples such as

$$P(x) = \frac{\pi^2}{4}\,,\ \frac{1}{(1-x^2)^2}\,,\ \frac{2}{1-x^2}\,,$$

can be obtained from conditions for univalence of analytic functions in the disk $\mathbb{D} = \{|z| < 1\}$. For these choices the criteria $|Sf(z)| \leq 2P(|z|)$ in $\mathbb{D}$ admit extremal functions that are unique up to Möbius transformations and which map the interval $[-1, 1]$ onto a closed curve. We shall show that no new extremal functions appear for these criteria in the context of curves in $\mathbb{R}^n$. Although not necessary, to make the discussion of this point as simple as possible, we will assume that $P(x)$ is an even function. This implies that the solution U_0 of $U'' + PU = 0$ with initial conditions $U_0(0) = 1, U_0'(0) = 0$ is also even, and hence can have no zeros on $(-1, 1)$ since otherwise it would have at least two. We define

$$F(x) = \int_0^x U_0^{-2}(t)dt\,,$$

so that F is odd and satisfies $SF = 2P$, $F(0) = 0, F'(0) = 1, F''(0) = 0$. When we regard F as a mapping of $(-1, 1)$ into $\mathbb{R} \subset \mathbb{R}^n \cup \{\infty\}$, the mappings $T \circ F$ with T Möbius are precisely those that manifest extremal behavior. More precisely, we have

Theorem D: *Let $f : (-1, 1) \to \mathbb{R}^n \cup \{\infty\}$ satisfy $f(0) = 0$, $|f'(0)| = 1, f''(0) = 0$ and suppose that $S_1 f(x) \leq 2P(x)$. Let P be as in Theorem B, and in addition be even. Then*

(a) $|f'(x)| \leq F'(|x|)$ on $(-1, 1)$ and f admits a (spherically) continuous extension to $[-1, 1]$.

(b) If $F(1) < \infty$, then f is one-to-one on $[-1, 1]$ and $f([-1, 1])$ has finite length.

(c) If $F(1) = \infty$, then either f is one-to-one on $[-1, 1]$ or, up to rotation, $f = F$.

Proof: It is not difficult to see that the normalization assumed in the statement can always be achieved by taking a suitable Möbius transformation of f. Indeed, if we map the osculating sphere of f at $f(0)$ onto a 2-dimensional subspace $\mathbb{R}^2$ (regarded as $\mathbb{C}$) with a Möbius transformation T, then, after replacing f by $T \circ f$, we can regard $f(0), f'(0)$, and $f''(0)$ as complex numbers. After suitable translation, rotation and dilation we can then obtain $f(0) = 0, f'(0) = 1$, and $f''(0) = 2\alpha$. Composition of the extension to $\mathbb{R}^n$ of the Möbius map $z/(1 + \alpha z)$ of the plane with this f results in one with the desired properties. Let again $v = |f'|^{-1/2}$. As pointed out in the proof of Theorem B, $v'' + qv = 0$ for some $q \leq P$, and because of the normalization of f, $v(0) = 1, v'(0) = 0$. Thus the Sturm comparison theorem implies that $v(x) \geq U_0(x)$, so that $|f'(x)| \leq F'(|x|)$.

If $F(1) < \infty$ then both integrals

$$\int_0^1 |f'(x)|dx \quad , \quad \int_{-1}^0 |f'(x)|dx$$

are finite, which implies that f admits a continuous extension to $[-1,1]$ and that $f([-1,1])$ has finite length.

Suppose that $F(1)=\infty$, and let $G(y)=F^{-1}(y)$, $-\infty<y<\infty$. We consider the function

$$w(y)=(\frac{v}{U_0})(G(y))\,. \tag{3.2}$$

Since $G'(y)=U_0^2(G(y))$, it follows easily that

$$w''=(P-q)U_0^4 w\,,$$

where P,q,U_0 are evaluated at $G(y)$. Also, $w(0)=1, w'(0)=0$. Because $2q\le S_1 f\le 2P$, w is convex. We claim that on each of the half-intervals $(-1,0]$ and $[0,1)$, either $f=F$ (up to rotation), or else, f can be extended to the endpoint so that the image of that half has finite length. The analysis being the same for each half, we consider $[0,1)$. If $q<P$ at a single point, then $w(y)\ge ay+b, a>0$ for all large y. Hence for x close to 1

$$|f'(x)|=v^{-2}(x)\le\frac{U_0^{-2}(x)}{(aF(x)+b)^2}=\frac{F'(x)}{(aF(x)+b)^2}=-\frac{1}{a}\frac{d}{dx}\left(\frac{1}{aF(x)+b}\right)\,, \tag{3.3}$$

which implies that $\int_0^1|f'|dx<\infty$, so that $f([0,1))$ once again has finite length, and f admits a continuous extension to $[0,1]$. On the other hand, it follows from (3.1) that $q\equiv P$ on $[0,1)$ only if $S_1 f=2P$ and f',f'' are linearly dependent. But then f maps that half onto a line, and because of the normalization at the origin and the fact that $S_1 f=P$ it follows that, up to a rotation, $f=F$, and again we have a spherically continuous extension. This completes the proof of (a).

It remains only to show that this continuous extension to $[-1,1]$ is injective except in the case of (c) when f coincides with the extremal F on the entire interval. If f is not one-to-one, then either $f(1)=f(-1)$ or there exists $x_0\in(-1,1)$ such that $f(x_0)$ equals, say $f(1)$ (the case $f(x_0)=f(-1)$ being the same except for notational details). Thus, in either case there exists $x_0\in[-1,1)$ such that $f(x_0)=f(1)$ and f is one-to-one on $[x_0,1)$. Let T once again be a Möbius transformation such that $g=T\circ f$ satisfies $g(1)=\infty$. Then $v=|g'|^{-1/2}$ is regular on $(x_0,1)$ and satisfies $v''+qv=0$, where $2q\le S_1 f\le 2P$ as in (3.1). It is easily verified that the general solution of $U''+PU=0$ is $\alpha U_0+\beta U_0 F=(\alpha+\beta F)U_0$. Let $c=(1+x_0)/2$. If we choose a,b such that $v(c),v'(c)$ coincide with the corresponding values for $(a+bF)U_0$, then by Sturm comparison, $v\ge(a+bF)U_0$ on any subinterval of $(x_0,1)$ containing c on which $(a+bF)U_0$ is positive. Since F is increasing and $a+bF(c)=v(c)/U_0(c)>0$, $a+bF$ will have to be positive on at least one of (x_0,c) or $(c,1)$. Then on this interval

$$|g'|\le\frac{1}{(a+bF)^2U_0^2}=\frac{F'}{(a+bF)^2}\,,$$

so that we will have $\int_c^1|g'|dx<\infty$ or $\int_{x_0}^c|g'|dx<\infty$ (contradicting of the fact that $g(x_0)=g(1)=\infty$), unless $b=0, x_0=-1$ and $F(1)=F(-1)=\infty$. Since in this case $g(-1)=g(1)=\infty$, we can replace g by a multiple of it so that $v(0)=1$ (and $v'(0)=0$). We consider again the convex function w defined in (3.2), and recall that the analysis leading to (3.3) shows that g cannot be infinite at both 1 and -1 unless $S_1 g=2P$ and $g((-1,1))$ is a straight line. Because $g=T\circ f$ and $f(0)=0, |f'(0)|=F'(0), f''(0)=F''(0)$ it is clear that f is a rotation of F.

4. FINAL COMMENTS

1. The situation considered in part (b) of Theorem D is essentially the case of a nonsharp univalence criterion. More precisely, it can be shown in this case that when $(1-x^2)^2P(x)$ is nonincreasing

then there exists $\lambda > 1$ such that $S_1 f \leq 2\lambda P$ still implies injectivity [Ch]. We also point out that Theorem D is a curve analogue of a theorem of Gehring and Pommerenke [Ge-Po].

2. The Schwarzian for curves as presented in Section 2 makes sense for C^3 curves in a Hilbert space of arbitrary dimension, since the osculating sphere remains meaningful in that context. Indeed, the normalizing procedures as well as the inversion operation taking a point to infinity, used in the proofs are well-defined and continue to leave the Schwarzian unaltered. For this reason, Theorems A,B,C,D carry over verbatim.

3. Since, as indicated in the introduction, injectivity for curves based on bounds on Sf translates into injectivity for conformal mappings, Theorem B should have a counterpart for $F : D \to \mathbb{R}^n$, for appropriate domains $D \subset \mathbb{R}^n$, and indeed it does, if one is content with bounds on SF calculated in all directions. It is easy to see, for example, how this would work for convex D, in which case an optimal bound would be $2\pi^2/(\operatorname{diam} D)^2$. It would be nice, however, to find a more elegant statement to this effect, based perhaps on a bound for a single expression involving partial derivatives of order up to 3 of F.

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